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Non-Abelian Duality for Open Strings¹

S. Förste^{§ 2}, A.A. Kehagias^{# 3} and S. Schwager^{§ 4}

[§] Sektion Physik
 Universität München
 Theresienstraße 37, 80333 München
 Germany

[#] Physik Department
 Technische Universität München
 D-85748 Garching, Germany

Abstract

We examine non-abelian duality transformations in the open string case. After gauging the isometries of the target space and developing the general formalism, we study in details the duals of target spaces with $SO(N)$ isometries which, for the $SO(2)$ case, reduces to the known abelian T-duals. We apply the formalism to electrically and magnetically charged 4D black hole solutions and, as in the abelian case, dual coordinates satisfy Dirichlet conditions.

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² E-Mail:Stefan.Foerste@physik.uni-muenchen.de

³E-Mail: kehagias@physik.tu-muenchen.de

⁴E-Mail:Stefan.Schwager@physik.uni-muenchen.de

1 Introduction

According to the duality assertion, a string theory compactified on a manifold K is equivalent to another theory (possibly the same) compactified on the dual manifold \bar{K} . Duality symmetries were first realized in the spectrum of the closed bosonic string [1]. A bosonic string compactified on a circle of radius R is equivalent to the theory obtained if the string was compactified on a circle of radius $1/R$ provided that momentum and winding modes are interchanged. In the case of closed superstrings it was proven later that T-duality is not, strictly speaking, a symmetry but rather it maps the type IIA theory to type IIB and vice versa [2],[3]. It was subsequently realized that the same duality symmetry also exists in the case of non-trivial backgrounds with abelian isometries [4]–[11]. Similarly, in this case although T-duality is a symmetry of the bosonic string, it is not of the supersymmetric one and again type IIA theory is mapped to type IIB and vice versa [11].

Duality transformations may also be discussed for target spaces with non-abelian isometries [12]–[16]. The general feature in this case is that the dual space has in general no isometries indicating that duality transformations may not necessarily be related with isometries of the target space but rather may have different origin. At this point one should also recall the duality-like symmetries of Calabi-Yau spaces [17].

The general procedure is the same as in the abelian case [4],[6]. Namely, one starts by gauging the isometries of the target space in the σ -model action and imposing the constraint, by means of a Lagrange multiplier, that the corresponding field strength vanishes. Integrating out the Lagrange multiplier one then obtains the original σ -model action one started with. Integrating out the gauge field, the dual theory is obtained which is again described by a σ -model action but with a different, even topologically, target space. This procedure has been proven to be equivalent to the first order formulation in the abelian as well as in the non-abelian case. The difference is that in the former case the dual and the original theory are equivalent as CFT [6], after considering some global aspects of the procedure, while in the later case they are not [14]. It should also be noted, that there exists the dilaton shift [18] compensating for the different “number” of integrations needed to obtain the dual and the original theory.

In the open string case now, things are quite different [3],[19],[20]. Type I theory for example compactified on a circle of radius R does not have a T-duality symmetry. However, one can still map the theory by a T-duality transformation to another alternative type I' theory in trying to understand its small R behavior [20]. This type I' theory has extended dynamical objects, the D-branes [3], which are the carriers of the R-R charges [21]. These results may also be extended to the case of non-trivial backgrounds [22],[23],[25]. Here, we

will consider the general case in which the target has non-abelian isometries.

In the next section and in order to establish notation and conventions we show how one may gauge the isometries of the open string. In sect. 3 we discuss the non-abelian duality for open strings in general non-trivial backgrounds with U(1) charges attached at the ends of the strings. We also give explicitly the duality transformations for target spaces with SO(N) isometry group which we employ in sect. 4 for some 4D black-hole solutions. In sect. 5 we discuss our results and we make some concluding remarks.

2 Gauging the isometries of the open string

We will consider here open strings described by the σ -model action

$$S = -\frac{1}{2} \int_{\Sigma} d^2\sigma \left(G_{mn} \partial_a X^m \partial^a X^n + B_{mn} \epsilon^{ab} \partial_a X^m \partial_b X^n \right) - \int_{\partial\Sigma} ds A_m \partial_s X^m + \frac{1}{4\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} R^{(2)} \Phi - \frac{1}{2\pi} \int_{\partial\Sigma} ds k \hat{\Phi}, \quad (1)$$

where we have parametrized the boundary of the world sheet Σ by s and ∂_s is the tangent derivative. The fields X^m , ($m, n = 1, \dots, N$) are coordinates of the N-dimensional target space M with metric and antisymmetric fields G_{mn} , B_{mn} , respectively, and A_m is a U(1) gauge field with field strength F_{mn} . We have also included a dilaton Φ ($\hat{\Phi}$) in the bulk (boundary) of the string where $R^{(2)}$ is the scalar curvature of Σ and k is the geodesic curvature of the boundary $\partial\Sigma$.⁵

We will assume now that M has some isometries, generated by a set of Killing vectors $\xi_I = \xi_I^m \partial_m$, ($I, J = 1, \dots, D$) which form the algebra of a D-dimensional group G with structure constants f_{IJ}^K , i.e.,

$$[\xi_I, \xi_J] = f_{IJ}^K \xi_K. \quad (2)$$

Then, under the action of the group, the coordinates X^m transform as

$$\delta X^m = \epsilon^I \xi_I^m. \quad (3)$$

In the case of neutral strings (with opposite charges attached at their ends), we may write the integral over the boundary as an integral in the bulk by simply replacing B_{mn} by $B_{mn} - F_{mn}$. In this case, the action is invariant under (3) if

$$\mathcal{L}_{\xi_I} G_{mn} = 0$$

⁵From the string theory point of view there is no difference in Φ and $\hat{\Phi}$ since the dilaton couples to the Gauss-Bonnet density of the world sheet.

$$\begin{aligned}
\mathcal{L}_{\xi_I}(B_{mn} - F_{mn}) &= 0 \\
\mathcal{L}_{\xi_I}\Phi &= 0 \\
\mathcal{L}_{\xi_I}\hat{\Phi} &= 0.
\end{aligned} \tag{4}$$

Recalling that the antisymmetric and the U(1) fields are defined only up to exact two- and one-forms, respectively, we get

$$\begin{aligned}
\mathcal{L}_{\xi_I}B_{mn} &= \partial_m\omega_{In} - \partial_n\omega_{Im}, \\
\mathcal{L}_{\xi_I}A_m &= \partial_m\phi_I + \omega_{Im},
\end{aligned} \tag{5}$$

for some zero- and one-forms ϕ^I , ω_I , respectively [27]. The rhs of (5) is invariant under the following transformations

$$\begin{aligned}
\omega_{Im} &\rightarrow \omega'_{Im} = \omega_{Im} + \partial_m h_I \\
\phi_I &\rightarrow \phi'_I = \phi_I - h_I + k_I,
\end{aligned} \tag{6}$$

where the k_I are constants and the h_I are scalars. By evaluating $\mathcal{L}_{[\xi_I, \xi_J]}B_{\mu\nu}$ and $\mathcal{L}_{[\xi_I, \xi_J]}A_m$ we find that ω_I and ϕ_I must satisfy the consistency conditions

$$\begin{aligned}
\mathcal{L}_{\xi_I}\omega_J - \mathcal{L}_{\xi_J}\omega_I &= f_{IJ}^K \omega_K + v_{IJ}, \\
\mathcal{L}_{\xi_I}\phi_J - \mathcal{L}_{\xi_J}\phi_I &= f_{IJ}^K \phi_K - \rho_{IJ},
\end{aligned} \tag{7}$$

where v_{IJ} is a closed one-form, i.e. it can locally be expressed as

$$v_{IJ} = d\rho_{IJ}.$$

We know that for an anomaly free gauged model in the closed string case to exist one should be able to remove v_{IJ} from (7) [24]. This implies certain integrability conditions on the v_{IJ} . Assuming that these conditions are satisfied we perform a transformation (6) to remove v_{IJ} from the first equation in (7) and then the consistency conditions read

$$\begin{aligned}
\mathcal{L}_{\xi_I}\omega_J - \mathcal{L}_{\xi_J}\omega_I &= f_{IJ}^K \omega_K, \\
\mathcal{L}_{\xi_I}\phi_J - \mathcal{L}_{\xi_J}\phi_I &= f_{IJ}^K \phi_K - k_{IJ},
\end{aligned} \tag{8}$$

with k_{IJ} being constants. Now, the residual symmetries of (6) are just constant shifts in ϕ_I under which the k_{IJ} will transform as

$$k_{IJ} \rightarrow k_{IJ} + f_{IJ}^K k_K. \tag{9}$$

As we will see later on the constants k_{IJ} have to satisfy certain integrability conditions.

Let us now gauge the isometries by assuming that the group parameters ϵ^I in eq.(3) are local. Then, following [24], we introduce gauge fields Ω_a^I on the world sheet and we demand invariance of the action under

$$\begin{aligned}\delta X^m &= \epsilon^I \xi_I^m, \\ \delta \Omega_a^I &= \partial_a \epsilon^I + f_{KJ}^I \Omega_a^K \epsilon^J.\end{aligned}\tag{10}$$

We define covariant derivatives by

$$D_a X^m = \partial_a X^m - \xi_I^m \Omega_a^I,\tag{11}$$

such that $D_a X^m$ transforms as

$$\delta(D_a X^m) = \epsilon^I \partial_n \xi_I^m D_a X^n.\tag{12}$$

The method to construct a gauge invariant action in the closed string case is discussed in [24]. For general B fields the gauge invariant action is not given by replacing partial derivatives with covariant ones. Since here we are mainly interested in modifications due to the boundary we put from now on the B field to zero and hence minimal coupling works for the bulk part of the action,

$$S_{bulk} = -\frac{1}{2} \int_{\Sigma} d^2 \sigma G_{mn} D_a X^m D^a X^n,\tag{13}$$

where dilaton contributions have been omitted.

By using the identity $\mathcal{L}_{\xi} = \iota_{\xi} d + d \iota_{\xi}$ where ι_{ξ} is the inner product in the exterior algebra, we get that the variation of the action (1) under the global transformations eq.(3) is

$$\delta S = - \int d^2 \sigma \epsilon^I \mathcal{L}_{\xi_I} G_{mn} \partial_a X^m \partial^a X^n - \int ds \epsilon^I \xi_I^m F_{mn} \partial_s X^n.\tag{14}$$

Thus, (1) is invariant if $\mathcal{L}_{\xi_I} G_{mn} = 0$ and if, in addition, there exists an exact one-form dv_I such that

$$\xi_I^m F_{mn} = \partial_n v_I.\tag{15}$$

Eq.(15) implies that $\mathcal{L}_{\xi_I} F_{mn} = 0$ so that

$$\mathcal{L}_{\xi_I} A_m = \partial_m \phi_I,\tag{16}$$

in agreement with eq.(5). By using eqs.(15,16), we get

$$\begin{aligned}\xi_I^m F_{mn} &= \xi_I^m \partial_m A_n - \xi_I^m \partial_n A_m = -\partial_n \phi_I - A_m \partial_n \xi_I^m - \xi_I^m \partial_n A_m \\ &= \partial_n (\phi_I - A_m \xi_m^I) = \partial_n v_I,\end{aligned}\tag{17}$$

so that ϕ_I and v_I are not independent but, up to a constant,

$$A_m \xi_I^m - \phi_I = v_I. \quad (18)$$

In order to find the gauge invariant action we may follow Noether's procedure, i.e., we first calculate the variation of the ungauged action and then by adding terms linear in the gauge field we try to cancel these variations. If $S^{(0)} = \int ds A_m \partial_s X^m$,

$$\delta S^{(0)} = \int_{\partial\Sigma} ds \left(\epsilon^I \mathcal{L}_{\xi_I} A_m \partial_s X^m + A_m \xi_I^m \partial_k \epsilon^I \partial_s X^k \right) \quad (19)$$

and one may try to cancel $\delta S^{(0)}$ by adding the term $S^{(1)} = \int ds C_I \Omega_s^I$ to $S^{(0)}$ where Ω_s is the component of the gauge field in the tangent direction. Then

$$\delta S^{(1)} = \int_{\partial\Sigma} ds \left(\partial_m C_I \Omega_s^I \epsilon^J \xi_J^m + C_I \partial_n \epsilon^I \partial_s X^n + C_I f_{JK}^I \Omega_s^J \epsilon^K \right), \quad (20)$$

and thus,

$$\begin{aligned} \delta S^{(0)} + \delta S^{(1)} = & \int_{\partial\Sigma} ds \left(\epsilon^I \partial_m \phi_I \partial_s X^m + A_m \xi_I^m \partial_k \epsilon^I \partial_s X^k + \epsilon^I \xi_I^m \partial_m C_J \Omega_s^J + \right. \\ & \left. C_I \partial_m \epsilon^I \partial_s X^m + C_I f_{JK}^I \Omega_s^J \epsilon^K \right). \end{aligned} \quad (21)$$

The condition for gauge invariance turns then out to be

$$C_I = -A_m \xi_I^m + \phi_I + \lambda_I \quad (22)$$

$$\mathcal{L}_{\xi_I} C_J = -f_{JI}^K C_K, \quad (23)$$

where λ_I are constants. It is not difficult to verify that (23) is compatible with (22) if the following condition is satisfied

$$k_{IJ} - f_{IJ}^K \lambda_K = 0, \quad (24)$$

where k_{IJ} is defined in (8). The integrability condition of (24) follows from the Jacobi identities of the isometry algebra and reads

$$f_{IJ}^K k_{KL} + \text{cycl. perm.} = 0. \quad (25)$$

The λ_K are not uniquely fixed but can be shifted according to (9). As we will see later the freedom of shifting λ_K represents the freedom in the position of the D-brane in the dual model. Thus, by using eq.(18), the gauge invariant action is

$$S = -\frac{1}{2} \int_{\Sigma} d^2\sigma G_{mn} D_a X^m D^a X^n - \int ds \left(A_m \partial_s X^m - v_I \Omega_s^I \right). \quad (26)$$

It should be noted that in the case of unoriented open strings (where $B=0$) the gauged action is again given by eq.(26). It is also obvious that the gauged action still has the $U(1)$ symmetry $A_m \rightarrow A_m + \partial_m \varphi$ of the original ungauged one.

3 Non-abelian T-duality

Below we examine T-duality transformations for open strings. For this, we will consider a σ -model with vanishing antisymmetric tensor since its presence does not modify the discussion and moreover it is absent in the examples we will consider later on. Then, in null-coordinates $z = \frac{1}{\sqrt{2}}(\tau + \sigma)$, $\bar{z} = \frac{1}{\sqrt{2}}(\tau - \sigma)$ ($\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$), the σ -model action is

$$S = \int_{\Sigma} d^2 z G_{mn} \partial X^m \bar{\partial} X^n - \int_{\partial \Sigma} ds A_m \frac{dX^m}{ds}, \quad (27)$$

where the dilaton terms have been omitted. We may enlarge the target space by adding fields X^α , ($\alpha, \beta = 1, \dots, d$) which parametrize a space K so that the target space is locally $M \times K$. Following the procedure of the previous chapter, one may gauge the isometries of M (or some subgroup) and then the gauged action turns out to be

$$\begin{aligned} S(X, \Omega) = & \int_{\Sigma} d^2 z \left(G_{\alpha\beta} \partial X^\alpha \bar{\partial} X^\beta + G_{\alpha m} \partial X^\alpha \bar{D} X^m + G_{m\alpha} D X^m \bar{\partial} X^\alpha + G_{mn} D X^m \bar{D} X^n \right) \\ & + \int_{\Sigma} d^2 z \Lambda_I F^I - \int_{\partial \Sigma} ds \left(A_\alpha \partial_s X^\alpha + A_m \partial_s X^m - v_I \left(\Omega^I \frac{dz}{ds} + \bar{\Omega}^I \frac{d\bar{z}}{ds} \right) \right). \end{aligned} \quad (28)$$

We have also included above the term $\Lambda_I F^I$ where Λ_I is a Lagrange multiplier and $F = \partial \bar{\Omega} - \bar{\partial} \Omega + [\Omega, \bar{\Omega}]$ is the field strength of the gauge field Ω . It is then straightforward to verify that (28) is invariant under the local transformations

$$\begin{aligned} \delta X^m &= \epsilon^I \xi_I^m, \\ \delta \Omega^I &= \partial \epsilon^I + f_{JK}^I \Omega^J \epsilon^K, \\ \delta \bar{\Omega}^I &= \bar{\partial} \epsilon^I + f_{JK}^I \bar{\Omega}^J \epsilon^K, \\ \delta \Lambda_I &= -f_{IJ}^K \Lambda_K \epsilon^J. \end{aligned} \quad (29)$$

The path integral now for the open string is as usual an integral over all field configurations which, however, are consistent with some specified boundary conditions

$$Z(\{S_B\}) = \int_{\{S_B\}} DX e^{iS(X)}. \quad (30)$$

and it is a functional of the boundary conditions, collectively written as $\{S_B\}$. The total vacuum to vacuum amplitude is then given by an integration of the $Z(\{S_B\})$'s over all allowed boundary conditions. The path integral after gauging is

$$\tilde{Z}(\{S_B\}) = \int_{\{S_B\}} DX \frac{D\Omega D\bar{\Omega}}{\mathcal{V}} D\Lambda e^{iS(X, \Omega)} \quad (31)$$

where we have divided out the “volume” \mathcal{V} of the gauge group. Integration over Λ constrains the gauge fields to be flat, $\Omega = h^{-1}dh$ for some $h \in G$. Fixing the gauge by choosing a section $h(z, \bar{z})$ gives back the ungauged action $S(X')$ with transformed coordinates $X' = hX$. However, the boundary conditions may have changed. The requirement that a pure gauge does not modify them will give boundary conditions for the gauge parameters which can be implemented by a second Lagrange multiplier. For Neumann conditions, for example, one has to restrict the normal component of the gauge field to vanish at the boundary. We will not do that here since the final result does not depend on this second Lagrange multiplier as one can check by an explicit calculation.

Our next task is to integrate over the gauge fields Ω . For this we perform a partial integration in the term $\int d^2z \Lambda_I F^I$ and after some algebra, the complete action is found to be

$$\begin{aligned}
S_g(X, \Omega) = & \int_{\Sigma} d^2\sigma \left(G_{\alpha\beta} \partial X^\alpha \bar{\partial} X^\beta + G_{\alpha m} \partial X^\alpha \bar{\partial} X^m + G_{m\alpha} \partial X^m \bar{\partial} X^\alpha + \right. \\
& G_{mn} \partial X^m \bar{\partial} X^n - \Omega^I \bar{h}_I - \bar{\Omega}^I h_I + \Omega^I f_{IJ} \bar{\Omega}^J \Big) - \int_{\partial\Sigma} ds \left(A_\alpha \frac{dX^\alpha}{ds} + \right. \\
& \left. A_m \partial_s X^m - v_I \left(\Omega^I \frac{dz}{ds} + \bar{\Omega}^I \frac{d\bar{z}}{ds} \right) - \Lambda_I \Omega^I \frac{dz}{ds} - \Lambda_I \bar{\Omega}^I \frac{d\bar{z}}{ds} \right), \tag{32}
\end{aligned}$$

where h_I, \bar{h}_I and f_{IJ} are given by

$$\begin{aligned}
h_I &= \partial \Lambda_I + \partial X^m G_{mn} \xi_I^n + \partial X^\alpha G_{\alpha m} \xi_I^m, \\
\bar{h}_I &= -\bar{\partial} \Lambda_I + \bar{\partial} X^m G_{nm} \xi_I^n + \bar{\partial} X^\alpha G_{m\alpha} \xi_I^m, \\
f_{IJ} &= f_{IJ}{}^K \Lambda_K + G_{mn} \xi_I^m \xi_J^n. \tag{33}
\end{aligned}$$

As a result, the path integral is

$$\begin{aligned}
Z = \int DX \frac{D\Omega D\bar{\Omega}}{\mathcal{V}} D\Lambda \exp i \Big[S + \int_{\Sigma} d^2z \left(-\Omega^I \bar{h}_I - \bar{\Omega}^I h_I + \Omega^I f_{IJ} \bar{\Omega}^J \right) - \\
\int_{\partial\Sigma} ds \left((\Omega^I \frac{dz}{ds} + \bar{\Omega}^I \frac{d\bar{z}}{ds}) (\Lambda_I + v_I) \right) \Big], \tag{34}
\end{aligned}$$

where S is the ungauged action in eq.(27). In order to get the dual action we have to integrate out the gauge fields $\Omega^I, \bar{\Omega}^I$. Here, however, due to the boundary term in the exponent, one has to integrate the gauge field independently in the bulk and on the boundary [25]. The integration in the bulk gives the dual action [13]

$$\tilde{S}[X, \Lambda] = S[X] - \int d^2\sigma h_I (f^{-1})^{IJ} \bar{h}_J, \tag{35}$$

where $S[X]$ is the Λ -independent part. Since the gauge field appears linearly in the boundary term (the last term in eq.(34)), its integration will produce a delta function [25], i.e., the constraints

$$\Lambda_I + v_I = 0 \quad \text{on } \partial\Sigma. \quad (36)$$

As a result, the dual fields have to satisfy Dirichlet conditions, no matter what boundary conditions the original fields satisfy. This means that in the dual theory the boundary conditions are imposed as a constraint irrespectively of any stationary conditions of the action.

As follows from eq.(23) v_I transforms in the same representation of the isometry group as the Λ_I so that the boundary condition eq.(36) is covariant. Hence, the position of the D-brane is gauge dependent. Moreover, as was mentioned before, v_I is defined only up to a constant corresponding to the U(1) symmetry of the original model. In the dual theory this symmetry manifests itself as shifts in the position of the D-brane.

Although it is not possible to give general formulas like in the abelian case, one has to discuss specific examples in order to illustrate the method. Below we will give the calculation for the SO(N) case since we will use it later to recover the abelian case and to find the duals of 4D black holes in the open string case.

As a specific example, we will assume that the target space is globally of the form $K \times S^{N-1}$ with the product metric

$$ds^2 = G_{\alpha\beta}(Y)dY^\alpha dY^\beta + V(Y)G_{ij}(\theta^i)d\theta^i d\theta^j, \quad (37)$$

where Y^α are coordinates on K . The metric G_{ij} of the coset $S^{N-1} = SO(N)/SO(N-1)$ is Y^α independent. One may embed S^{N-1} into R^N [13] and use Cartesian coordinates X^m instead of angular θ^i . The action is then

$$\begin{aligned} S[Y, X] = & S[Y] + \int d^2z V \left(G_{mn} \partial X^m \bar{\partial} X^n + \frac{1}{2\sqrt{V}R} \beta (G_{mn} X^m X^n - R^2) \right) - \\ & \int_{\partial\Sigma} ds A_m \frac{dX^m}{ds}, \end{aligned} \quad (38)$$

where $S[Y] = \int d^2\sigma G_{\alpha\beta} \partial Y^\alpha \bar{\partial} Y^\beta - \int ds A_\beta dY^\beta/ds$ and the Lagrange multiplier β imposes the constraint $X^2 = R^2$ so that it defines S^{N-1} of radius R. Gauging this action and fixing the gauge $\Omega = \bar{\Omega} = 0$ we get the original model with target-space metric (37). To find the dual action we have to fix a gauge. A conventional such choice is to take all but one X^m to vanish, i.e.,

$$X^m = 0, m = 1, \dots, N-1, X^N = R. \quad (39)$$

Of course this choice does not completely fix the gauge freedom since we still have $SO(N-1)$ invariance. We may use the remaining $SO(N-1)$ gauge freedom to gauge away $1/2(N-1)(N-2)$ of the dual coordinates Λ_I .

Let us now apply the above procedure to the abelian case. Here the group is $SO(2)$ with the generator $T_{mn} = \epsilon_{mn}$. The Killing vector and the $SO(2)$ -invariant $U(1)$ gauge field are

$$\begin{aligned}\xi_m &= T_{mn}X^n = \epsilon_{mn}X^n, \\ A_m &= Q(Y)\epsilon_{mn}X^n,\end{aligned}\tag{40}$$

respectively. The corresponding action turns out to be

$$\begin{aligned}S[Y, X] &= S[Y] + \int d^2\sigma V \left(\delta_{mn} \partial X^m \bar{\partial} X^n + \frac{1}{2\sqrt{V}R} \beta (\delta_{mn} X^m X^n - R^2) \right) - \\ &\quad \int_{\partial\Sigma} ds Q \epsilon_{mn} X^n \frac{dX^m}{ds},\end{aligned}\tag{41}$$

By gauging this action, fixing the gauge $\Omega, \bar{\Omega} = 0$ and eliminating the Lagrange multiplier β we get

$$S[Y, X] = S[Y] + \int d^2\sigma R^2 V \partial\theta\bar{\partial}\theta - \int_{\partial\Sigma} ds Q \epsilon_{mn} X^n \frac{dX^m}{ds}.\tag{42}$$

To obtain the dual action, we first fix the gauge $X^1 = 0, X^2 = R$. Then, eq.(33) gives

$$h = \partial\Lambda, \quad \bar{h} = -\bar{\partial}\Lambda, \quad f = VR^2\tag{43}$$

and the dual action turns out to be

$$\tilde{S}[Y, \Lambda] = S[Y] + \int d^2\sigma \frac{1}{VR^2} \partial\Lambda\bar{\partial}\Lambda,\tag{44}$$

The dual field Λ satisfies the boundary condition (36) and since $\mathcal{L}_\xi A_m = \partial_m\phi = 0$ we have $v = QR^2 - v_0$ so that

$$\Lambda + QR^2 = v_0 \text{ on } \partial\Sigma.\tag{45}$$

4 Non-abelian duals of 4D black holes

We will apply here the formalism developed in the previous section to find the non-abelian duals of 4D black holes. The four-dimensional low-energy effective action for open strings [26],[27] and in the string frame is,

$$I_{eff} = \int d^4x \sqrt{G_\sigma} \left(e^{-2\Phi} \frac{1}{2} R + 2e^{-2\Phi} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{8} e^{-\Phi} F_{\mu\nu} F^{\mu\nu} \right),\tag{46}$$

where we have considered a U(1) gauge field only and no antisymmetric field. The different powers of $e^{-\Phi}$ in front of the gravitational and gauge terms appear because the former term arises from the sphere while the latter comes from the disc. The field equations in the Einstein frame $G_{\mu\nu} = e^{-2\Phi} G_{\mu\nu}^\sigma$ are

$$0 = R_{\mu\nu} - 2\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}e^{-\Phi}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{8}G_{\mu\nu}e^{-\Phi}F_{\rho\lambda}F^{\rho\lambda}, \quad (47)$$

$$0 = \nabla^2\Phi + \frac{1}{16}e^{-\Phi}F_{\mu\nu}F^{\mu\nu}, \quad (48)$$

$$0 = \nabla_\mu(e^{-\Phi}F^{\mu\nu}). \quad (49)$$

We are looking for static, spherically symmetric solutions to the equations above. There exist two such solutions, a magnetically and an electrically charged black hole and we will first discuss the former one.

An SO(3) symmetric U(1) field strength $F_{\mu\nu}$ is the pure magnetic Maxwell field

$$F_m = Q \sin\vartheta d\vartheta \wedge d\varphi, \quad (50)$$

where, due to the Dirac quantization condition, Q must be an integer multiple of 1/2. Similarly, an SO(3)-invariant ansatz for the four-dimensional metric is

$$ds^2 = -\lambda^2 dt^2 + \frac{dr^2}{\lambda^2} + \rho^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (51)$$

where λ, ρ are functions of r only. Then, the solutions of eqs(47–49) are [29],

$$\lambda^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{3/5}, \quad (52)$$

$$\rho = r \left(1 - \frac{r_-}{r}\right)^{1/5}, \quad (53)$$

$$e^{-\Phi_m} = \left(1 - \frac{r_-}{r}\right)^{2/5}. \quad (54)$$

These solutions are given in terms of two integration constants r_+, r_- . The mass M and the charge Q are given in terms of these constants as

$$M = \frac{1}{2}r_+ + \frac{3}{10}r_-, \quad (55)$$

$$Q = 4\left(\frac{1}{5}r_+r_-\right)^{1/2}. \quad (56)$$

To find an electric solution, one may use the electric-magnetic duality $F_{mn} \rightarrow {}^*F_{mn}$ which is a symmetry of the field equations (47–49). In particular, in order to convert the field equation (49) into the Bianchi identity, the duality rotation

$${}^*F^{mn} = \frac{1}{2}e^{-\Phi}\epsilon^{mnkl}F_{kl}, \quad (57)$$

has to be performed. Then, the remaining field equations are invariant if we also transform the dilaton as $\Phi \rightarrow -\Phi$. Thus, the electrically charged solution is

$$F_e = e^{-\Phi_m} \frac{Q}{\rho^2} dt \wedge dr = \frac{Q}{r^2} dt \wedge dr, \quad (58)$$

$$e^{\phi_e} = \left(1 - \frac{r_-}{r}\right)^{2/5}, \quad (59)$$

with the same metric (51,52,53).

In order to apply the duality transformation, one has to transform the solutions above into the string frame. In this frame, the metric is of the form

$$ds^2 = -\alpha^2 dt^2 + \frac{dr^2}{\beta^2} + \gamma^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (60)$$

The metric components (α, β, γ) are functions of r only and are explicitly given by

$$\alpha_m^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{-1/5}, \quad (61)$$

$$\beta_m^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{7/5}, \quad (62)$$

$$\gamma_m^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{-2/5}, \quad (63)$$

for the magnetic solution and by

$$\alpha_e^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{7/5}, \quad (64)$$

$$\beta_e^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{-1/5}, \quad (65)$$

$$\gamma_e^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{6/5}, \quad (66)$$

for the electric one. The U(1) field strength is not affected by the Weyl transformation when going from the Einstein frame to the string frame.

The solutions given above are clearly SO(3) invariant and one may try to find the dual solution. This can be done in two ways. We express the angular part of the metric in Cartesian coordinates (X^1, X^2, X^3) and we impose the constraint $\sum_{i=1}^3 X^{i2} = 1$. We then gauge the action in the way described before and we fix the gauge by choosing

$$X^1 = X^2 = 0, \quad X^3 = 1, \quad \Lambda_2 = 0. \quad (67)$$

In this gauge and by taking $(T_I)_n^m = \frac{1}{\sqrt{2}} \epsilon_{Inm}$ which are properly normalized and satisfy the SO(3) algebra with $c_{IJ}^K = \frac{1}{\sqrt{2}} \epsilon_{IJK}$ we find by employing eq.(33)

$$h_I = \begin{pmatrix} \partial \Lambda_1 & 0 & \partial \Lambda_3 \end{pmatrix} \quad (68)$$

$$\bar{h}_I = \begin{pmatrix} -\bar{\partial}\Lambda_1 & 0 & -\bar{\partial}\Lambda_3 \end{pmatrix} \quad (69)$$

$$f_{IJ} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\gamma^2 & -\Lambda_3 & 0 \\ \Lambda_3 & \frac{1}{\sqrt{2}}\gamma^2 & -\Lambda_1 \\ 0 & \Lambda_1 & 0 \end{pmatrix}. \quad (70)$$

By using the above expressions one may evaluate $h_I(f^{-1})^{IJ}\bar{h}_J$ which is found to be

$$h_I(f^{-1})^{IJ}\bar{h}_J = -\frac{1}{\gamma^2(x^2 - y^2)} \left(\gamma^4 \partial y \bar{\partial} y + x^2 \partial x \bar{\partial} x \right), \quad (71)$$

where, as in [13], we have defined $x^2 = 2(\Lambda_1^2 + \Lambda_3^2)$ and $y^2 = 2\Lambda_2^2$. Thus, the dual space has a metric given by

$$ds^2 = -\alpha^2 dt^2 + \frac{dr^2}{\beta^2} + \frac{1}{\gamma^2(x^2 - y^2)} \left(\gamma^4 dy^2 + x^2 dx^2 \right). \quad (72)$$

The same result can also be found by using angular coordinates (ϑ, φ) for S^2 instead of the Cartesian X^m . In these angular coordinates the Killing vectors are

$$\begin{aligned} \xi_1 &= \frac{1}{\sqrt{2}} (\cos \varphi \partial_{\vartheta} - \sin \varphi \cot \vartheta \partial_{\varphi}), \\ \xi_2 &= \frac{1}{\sqrt{2}} (\sin \varphi \partial_{\vartheta} + \cos \varphi \cot \vartheta \partial_{\varphi}), \\ \xi_3 &= \frac{1}{\sqrt{2}} \partial_{\varphi}. \end{aligned} \quad (73)$$

We may gauge the action in the way described in section 2, and we fix the gauge by choosing $\vartheta = \pi/2$, $\varphi = 0$, $\Lambda_1 - \Lambda_3 = 0$. After the gauge fixing, we find that h_I, \bar{h}_I, f_{IJ} are given by

$$h_I = \begin{pmatrix} \partial\Lambda_1 & \partial\Lambda_2 & -\partial\Lambda_1 \end{pmatrix} \quad (74)$$

$$\bar{h}_I = \begin{pmatrix} -\bar{\partial}\Lambda_1 & -\bar{\Lambda}_2 & \bar{\partial}\Lambda_1 \end{pmatrix} \quad (75)$$

$$f_{IJ} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\gamma^2 & -\Lambda_1 & -\Lambda_2 \\ \Lambda_1 & 0 & \Lambda_1 \\ \Lambda_2 & -\Lambda_1 & \frac{1}{\sqrt{2}}\gamma^2 \end{pmatrix}. \quad (76)$$

Proceeding as before, one may evaluate $h_I(f^{-1})^{IJ}\bar{h}_J$ and the dual metric, as expected, is again given by eq.(72) with $x^2 = 2(2\Lambda_1^2 + \Lambda_2^2)$ and $y^2 = 2\Lambda_2^2$ this time.

Let us now turn to the boundary conditions. In the electric case, there is no U(1) field in the (ϑ, φ) direction and the dual coordinates (x, y) are restricted to satisfy the Dirichlet conditions

$$x = x_0, \quad y = y_0 \quad \text{on } \partial\Sigma, \quad (77)$$

where (x_0, y_0) are constants. For the magnetic case, the gauge potential in the upper hemisphere is $A = Q(1 - \cos \vartheta)d\varphi$. Then, by employing eq.(16) we find

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}} \sin \varphi \left(\frac{1}{\sin \vartheta} - \cot \vartheta \right), \\ \phi_2 &= -\frac{1}{\sqrt{2}} \cos \varphi \left(\frac{1}{\sin \vartheta} - \cot \vartheta \right), \\ \phi_3 &= \phi_3^0,\end{aligned}\tag{78}$$

and from eq.(18) we get

$$\begin{aligned}v_1 &= \frac{Q}{\sqrt{2}} \frac{\sin \varphi}{\sin \vartheta} + v_1^0, \\ v_2 &= \frac{Q}{\sqrt{2}} \sin \varphi \sin \vartheta + v_2^0, \\ v_3 &= \frac{Q}{\sqrt{2}} (1 - \cos \vartheta) + v_3^0,\end{aligned}\tag{79}$$

where (v_1^0, v_2^0, v_3^0) are constants. After gauge fixing the dual fields turn out to satisfy the Dirichlet conditions (77). It should be noted that after the gauge fixing, there remain two independent parameters out of (v_1^0, v_2^0, v_3^0) which represents the freedom in the position of the D-brane.

5 Conclusions

We have examined here the non-abelian duality transformations for open strings following the path integral approach [6]. There exist some peculiarities due to the boundary [28]. Dirichlet boundary conditions in the dual model arise as in the abelian case. This shows that also for open strings moving in a background with non-abelian isometries D-branes appear in a natural way. It should be stressed that the Dirichlet conditions seem to be imposed in the dual model as external conditions irrespectively of any stationary condition of the action. This is because one integrates the gauge fields on the boundary independently of the gauge fields on the bulk [25] and it is this integration that produces the Dirichlet conditions.

We have applied the non-abelian duality transformation in the case of 4D target spaces. In particular we have discussed an electrically and a magnetically charged black hole. For both of these solutions we have presented their non-abelian duals. The dual coordinates satisfy Dirichlet conditions which define curved 1-branes. The world volume of these branes has black-hole structure as well. The singularity is at $r = r_-$, the horizon is at $r = r_+$

and for the extremal case $r_+ = r_-$ we have a naked singularity. Thus, the dual model contains extended objects, namely D-branes, which will act as a source for the closed string modes, the graviton and the dilaton [30]. Conformal invariance is then imposed by the Born-Infeld action for the D-brane [3] which, moreover, will also specify the dilaton shift. In our case, the dual electric and magnetic solutions describe fundamental and solitonic 1-branes, respectively, since in the former case we expect a source term due to a non-vanishing electric field, while such a term is absent in the latter case since the coupling to the U(1) gauge field vanishes due to Dirichlet conditions. A detailed discussion of these issues including the dilaton is under investigation.

In our discussion we did not include global issues. For non-abelian dualities this is an outstanding problem for the case of closed strings, too. A further topic to be addressed in a future work is the extension of non-abelian T-duality to the open superstring as well as the inclusion of Chan-Paton factors.

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